Fourier Transform Exercises

Computational Astrophysics

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$$f(\mathbf{x}) = \sum_k f_{\mathbf{k}} e^{i\mathbf{k}\mathbf{x}} \quad ; \quad \mathbf{k} \in \frac{2\pi}{L}\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \text{ where } n_{1,2,3} \in \{0, 1, ..., N_{1,2,3} - 1\}$$

$$f_{\mathbf{k}} = \frac{1}{L^3}\int_V f(\mathbf{x}) e^{-i\mathbf{k}\mathbf{x}} d^3x$$

$$f_{\mathbf{k}} = \frac{1}{N_1 N_2 N_3}\sum_{\mathbf{p}} f_{\mathbf{p}} e^{-i\mathbf{k}\mathbf{x}_{\mathbf{p}}} \quad ; \quad \mathbf{x}_{\mathbf{p}} = \frac{L}{N}\mathbf{p} = \frac{L}{N}\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \text{ where } p_{1,2,3} \in \{0, 1, ..., N_{1,2,3} - 1\}$$

1 Fourier Transform tests

Some functions in maths have an analytical Fourier Transform. For example $f(x) = \exp(-\pi x^2)$. The Fourier Transform of this function is $f_k = \exp(-\pi k^2)$. Verify this using the discrete formulas given above. The normalization of the formulas may need to be adjusted (the analytical solution is a Fourier Transform, while the formulas above are Discrete Fourier Transform).

In the following exercises, you may find using a module or package for FFT convenient (for example the FFT module in numpy or scipy). Before doing that, you may repeat this exercise with the FFT module and reproduce the same results. Some modules may use a different normalization, especially for the k vector, or different signs. The order of the transformed array is also usually different (check for example "fftshift"), because some modules may also use negative frequencies ($n \in \{-N/2, ..., N/2-1\}$) and put them after the positive frequencies.

2 Signal analysis

The Fourier Transform can be used to identify the dominant frequency contributions (modes) of a signal. Given the function $f(x) = A\sin(2\pi\cdot 2x) + B\sin(2\pi\cdot 4x)$, verify numerically that you reconstruct the correct frequency contributions of the signal. Start with $A = B = 1$ then change values and check the results. You can also try to change the domain of integration and the resolution (N).

3 Gibbs phenomenon

Consider the step function $f(x) = 2H(x/L) - 1$ of length $2L$ (from $-L$ to L) with the Heaviside step function $H(x) = (0 \text{ if } x < 0 \text{ else } 1)$. This function shows an interesting property in its Fourier Transform. Compute the Fourier Transform Coefficients and use them to reconstruct and plot the step function (by summing the first n terms). What do you observe near $x = 0$? What happens if you include less/more terms in the Fourier sum? (Hint: The effect is called "Gibbs-Phenomenon").

4 Fourier Transform and derivatives

Calculate the derivative of the following functions $e^{-x^2}, \sin(x), \frac{1}{x}, \cos(x)e^{-x^2}$ by first doing Fourier transform, then multiplying by ik and performing inverse Fourier transform. This comes from $\mathcal{F}\left(\frac{df}{dx}\right) =$

$ik\mathcal{F}(f)$. Compare with the analytical solution for the derivatives. Does this procedure work for all from the mentioned functions? For which of them? Can you explain why?

5 Fourier Transform and convolution

In this part, we will experience how convolution works and where it can be used in astrophysical simulations. We will work in 1D in this exercise. In this case, the convolution of two functions f and g can we written as $(f*g)(x) = \int f(y)g(x-y)dy$. Now, let us investigate the convolution within the two mass assignment scenarios: cloud in cell (CIC) and triangular shaped cloud (TSC). First, we define

$$\Pi(x) = \begin{cases} \frac{1}{2} \text{ if } |x| \leq \frac{1}{2} \\ 0 \text{ else} \end{cases}$$

$$\Delta(x) = \begin{cases} 1 - |x| \text{ if } |x| \leq 1 \\ 0 \text{ else} \end{cases} \quad .$$

In the context of mass assignment for N-body simulations, we have seen that the portion of mass of the particle placed at x_i that is assigned to a grid cell p with center at x_p can be found as a convolution of a shape function $S(x)$ and the window function

$$\Pi_p(x) = \Pi\left(\frac{x - x_p}{h}\right),$$

so

$$W_p(x_i) = \int_p \Pi_p(x) S(x - x_i) dx.$$

The shape functions are $S(x) = \frac{1}{h}\Pi\left(\frac{x}{h}\right)$ for the case of CIC and $S(x) = \frac{1}{h}\Delta\left(\frac{x}{h}\right)$ for TSC.

- Calculate analytically and/or with a numerical integration $W_p(x_i)$ as a function of x_i for a chosen grid cell p with width h and for both CIC and TSC scenarios.
- Consider a particle contained in the given cell. In which position does the particle's portion of mass reach the maximum? Which value does this maximum have? Do it for both CIC and TSC.
- Consider a particle NOT contained in the given cell. In which position does the particle's portion of mass reach the maximum? Which value does this maximum have? Do it for both CIC and TSC.

A nice property of convolution is that, in Fourier space, it turns into a multiplication, namely: $\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$.

- Take the TSC scenario. Calculate the Fourier transform of $W_p(x)$ and then Fourier transform of both of the functions $S(x)$ and $\Pi_p(x)$ and verify the validity of above statement by e.g. overplotting the results.

6 Poisson equation

6.1 Density profile

Let's try to calculate the 3D gravitational potential of some very well known shape, the uniform sphere. Let's consider a space where $x \in [-1, 1]$, $y \in [-1, 1]$, $z \in [-1, 1]$, built in an uniform 3D grid, and a distribution of density that is $\rho(r) = \begin{cases} 1 & (r < R_0) \\ 0 & (r \geq R_0) \end{cases}$ where $r = \sqrt{x^2 + y^2 + z^2}$ and R_0 can be chosen within 0 and 1. First, build and plot the density distribution.

6.2 Fourier Transform

Let's calculate the 3D Fourier Transform of the density distribution. Let's calculate also the k's and be careful about their normalization (the normalization of the density's FT is not important if you are using the same package to transform back, otherwise you should be more careful).

Let's consider now G=1 in the Poisson equation. We can calculate the FT of the potential by multiplying the FT of the density distribution by $-\frac{4\pi}{|k|^2}$. Let's do that and transform back to find the potential. Just set all transforms = 0 when $|k| = 0$.

6.3 Comparison with the analytical solution

The gravitational acceleration caused by this density distribution is know analytically and it is $\vec{a} = -\frac{4\pi}{3}\vec{r}\begin{cases} 1 & r < R_0 \\ \left|\frac{R_0}{r}\right|^3 & r \geq R_0 \end{cases}$, while the gravitational potential is $\Phi = C + \frac{4\pi}{6}R_0^2 \begin{cases} \left|\frac{r}{R_0}\right|^2 - 3 & r < R_0 \\ -2\left|\frac{R_0}{r}\right| & r \geq R_0 \end{cases}$ where C is a free constant.

Let's calculate the acceleration from the potential on the x-axis and compare it with the analytical solution. Are the two curves in agreement? What happens at the boundaries, especially when you increase R_0?